1. (i) (a) $(1+\mathrm{i})^2 = 1+2\mathrm{i}+\mathrm{i}^2$ *(M1)*

$= 2\mathrm{i}$ *(A1)*

[2 marks]

(b) $(1+\mathrm{i})^{4n}$

Let $\mathrm{P}(n)$ be the proposition: $(1+\mathrm{i})^{4n} = (-4)^n$

We must first show that $\mathrm{P}(1)$ is true.

$(1+\mathrm{i})^4 = \left((1+\mathrm{i})^2\right)^2 = (2\mathrm{i})^2$ *(M1)*

$= 4(\mathrm{i})^2 = (-4)^1$ *(A1)*

Next, assume that for some $k \in \mathbb{N}^+$
$\mathrm{P}(k)$ is true, then show that $\mathrm{P}(k+1)$ is true.
$\mathrm{P}(k): (1+\mathrm{i})^{4k} = (-4)^k$ *(C1)*

Now, $(1+\mathrm{i})^{4(k+1)} = (1+\mathrm{i})^{4k}\,(1+\mathrm{i})^4$

$= (-4)^k\,(-4)$ *(M1)*

$= (-4)^{k+1}$ *(A1)*

Therefore, by mathematical induction $\mathrm{P}(n)$ is true for all $n \in \mathbb{N}^+$ *(C1)*

[6 marks]

(c) $(1+\mathrm{i})^{32} = (1+\mathrm{i})^{4(8)} = (-4)^8$ *(M1)*

$= 65536$ *(A1)*

[2 marks]

(ii) (a) $z_1 = \dfrac{\sqrt{6}-\mathrm{i}\sqrt{2}}{2}$ $\quad |z_1| = \sqrt{\dfrac{6}{4}+\dfrac{2}{4}}$ *(M1)*

$= \sqrt{2}$ *(A1)*

$\arg z_1 = \arctan\left(\dfrac{-1}{\sqrt{3}}\right) = -\dfrac{\pi}{6}$ *(A1)*

Therefore, $z_1 = \sqrt{2}\left(\cos\left(-\dfrac{\pi}{6}\right) + \mathrm{i}\sin\left(-\dfrac{\pi}{6}\right)\right)$ *(C1)*

$z_2 = 1-\mathrm{i}$ $\quad |z_2| = \sqrt{1+1} = \sqrt{2}$

$\arg z_2 = \arctan(-1) = -\dfrac{\pi}{4}$ *(A1)*

$z_2 = \sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right) + \mathrm{i}\sin\left(-\dfrac{\pi}{4}\right)\right)$ *(C1)*

[6 marks]

(b) $$\frac{z_1}{z_2} = \frac{\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)}{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)}$$

$= 1\left(\cos\left(-\dfrac{\pi}{6}+\dfrac{\pi}{4}\right) + \mathrm{i}\sin\left(-\dfrac{\pi}{6}+\dfrac{\pi}{4}\right)\right)$ *(M2)*

$= \cos\dfrac{\pi}{12} + \mathrm{i}\sin\dfrac{\pi}{12}$ *(AG)*

[2 marks]

continued…

Question 1 (ii) continued

(c) $\frac{z_1}{z_2} = \left(\frac{\sqrt{6} - \mathrm{i}\sqrt{2}}{2}\right)\left(\frac{1}{1-\mathrm{i}}\right)\left(\frac{1+\mathrm{i}}{1+\mathrm{i}}\right)$ ***(M1)***

$= \frac{\sqrt{6} + \sqrt{2} + \mathrm{i}\left(\sqrt{6} - \sqrt{2}\right)}{4}$ ***(A1)***

Therefore, $a = \cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$ ***(A1)***

$b = \sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$ ***(A1)***

[4 marks]

Note:	**Some students may use the half-angle formulas. Answers will only differ in form.** $\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$ $\sin\frac{\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}$

Total [22 marks]

2. (a) $\vec{AB} = -\vec{i} - 3\vec{j} + \vec{k}$, $\vec{BC} = \vec{i} + \vec{j}$

(A2)

[2 marks]

(b) $$\vec{AB} \times \vec{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -3 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

(M1)

$$= -\vec{i} + \vec{j} + 2\vec{k}$$

(A2)

[3 marks]

(c) Area of $\Delta ABC = \frac{1}{2}\left| -\mathrm{i} + \vec{j} + 2\vec{k} \right|$ *(M1)*

$$= \frac{1}{2}\sqrt{1+1+4}$$

$$= \frac{\sqrt{6}}{2}$$

(A1)

[2 marks]

(d) A normal to the plane is given by $\vec{n} = \vec{AB} \times \vec{BC} = -\vec{i} + \vec{j} + 2\vec{k}$ *(M1)*

Therefore, the equation of the plane is of the form $-x + y + 2z = g$, and since the plane contains A, then $-1 + 2 + 2 = g \Rightarrow g = 3$. *(M1)*

Hence, an equation of the plane is $-x + y + 2z = 3$. *(A1)*

[3 marks]

(e) Vector $\vec{n}$ above is parallel to the required line.

Therefore, $x = 2 - t$ *(M1)*

$y = -1 + t$

$z = -6 + 2t$ *(A1)*

[2 marks]

(f) Distance of a point (x_0, y_0, z_0) from a plane $ax + by + cz + d = 0$ is given by

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(M1)

since $-x + y + 2z - 3 = 0$ and D is $(2, -1, -6)$ *(M1)*

then; distance $= \frac{|-2-1-12-3|}{\sqrt{1+1+4}} = \frac{18}{\sqrt{6}}$

$= 3\sqrt{6}$ *(A1)*

[3 marks]

(g) Unit vector in the direction of $\vec{n}$ is $\vec{e} = \frac{1}{|\vec{n}|} \times \vec{n}$ *(M1)*

$$= \frac{1}{\sqrt{6}}\left(-\vec{i} + \vec{j} + 2\vec{k}\right)$$

(A1)

($-\vec{e}$ is also acceptable)

[2 marks]

(h) Let H be the intersection of DE with the plane, then

$-2 + t + (-1 + t) + 2(-6 + 2t) = 3$ *(M1)*

$\Rightarrow \quad 6t = 18$

$t = 3$ *(A1)*

$\Rightarrow \mathrm{H}(-1, 2, 0)$

but H is the mid point of DE *(M1)*

$\Rightarrow \mathrm{E}(-4, 5, 6)$ *(A1)*

[4 marks]

Total [21 marks]

3. (a) $f_k(x) = x\ln x - kx$

$\Rightarrow f_k'(x) = \ln x + 1 - k$ *(M1)(A1)*

[2 marks]

(b) $f_0'(x) = \ln x + 1$

$f_0(x)$ increases where $f_0'(x) > 0$ *(M1)*

$\Rightarrow \ln x > -1 \Rightarrow x > \frac{1}{\text{e}}$ *(A1)*

[2 marks]

(c) (i) Stationary points happen where $f_k'(x) = 0$

$\Rightarrow \ln x = k - 1$ *(M1)*

$\Rightarrow x = \text{e}^{k-1}$ *(A1)*

(ii) x intercepts are where $f_k(x) = 0$

$\Rightarrow x\ln x - kx = 0$

$\Rightarrow x(\ln x - k) = 0$ *(M1)*

$\Rightarrow x = 0$ or $\ln x = k$

$\Rightarrow x = \text{e}^k$

$\Rightarrow (\text{e}^k, 0)$ *(A1)*

[4 marks]

(d) Area $= \int_0^{\text{e}^k} \left|(x\ln x - kx)\right| \text{d}x \ \left(= \int_0^{\text{e}^k} (kx - x\ln x)\,\text{d}x\right)$ *(M1)*

Integrate $x\ln x$ by parts.

$\int x\ln x \,\text{d}x = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,\text{d}x$ *(M1)*

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$ *(A1)*

$\Rightarrow$ Area $= \left|\left[\frac{x^2}{2}\ln x - \frac{x^2}{4} - \frac{kx^2}{2}\right]_0^{\text{e}^k}\right|$ *(M1)*

$= \frac{\text{e}^{2k}}{4}$ *(A1)*

Note: Given $x\ln x - kx = f_k(x) \approx 0$ when $x = 0$.

[5 marks]

(e) Gradient of the tangent at $A(\text{e}^k, 0), m$ is $f_k'(\text{e}^k) = \ln \text{e}^k + 1 - k$ *(M1)*

$= 1$

Therefore, an equation of the tangent is $y = x - \text{e}^k$. *(A1)*

[2 marks]

continued...

Question 3 continued

(f) The tangent forms a right angle triangle with the coordinate axes. The prependicular sides are each of length e^k. *(M1)*

Area of the triangle $= \frac{1}{2} \times e^k \times e^k = \frac{1}{2}e^{2k}$ *(A1)*

$\frac{1}{2}e^{2k} = 2\left(\frac{1}{4}e^{2k}\right)$ *i.e.* The area of the triangle is twice the area enclosed by the curve and the x-axis. *(AG)*

[2 marks]

(g) Since the x-intercepts are of the form $x_k = e^k$, for $k \in \mathbf{N}$ *(M1)*

then $x_{k+1} = e^{k+1}$

and $\frac{x_{k+1}}{x_k} = e$ *(A1)*

Therefore, the x-intercepts $x_0, x_1, \ldots x_k, \ldots$ form a geometric sequence with $x_0 = 1$ and a common ratio of e. *(R1)*

[3 marks]

Total [20 marks]

4. (a) For $f(x)$ to be a probability distribution function, $\int_0^1 f(x)\,dx = 1$.

$\Rightarrow \int_0^1 (e - ke^{kx})\,dx = 1$ *(M1)*

$\Rightarrow \left[ex - e^{kx}\right]_0^1 = 1$ *(M1)*

$\Rightarrow e - e^k + 1 = 1$ *(A1)*

$\Rightarrow e = e^k \Rightarrow k = 1$ *(AG)*

Thus $f(x) = e - e^x, 0 \le x \le 1$

[3 marks]

(b) $\int_{1/4}^{1/2} \left(e - e^x\right)dx = \left[ex - e^x\right]_{1/4}^{1/2} = \frac{e}{2} - \sqrt{e} - \frac{e}{4} + \sqrt[4]{e}$ *(M1)*

$= \frac{e}{4} - \sqrt{e} + \sqrt[4]{e}$ *(A1)*

[2 marks]

(c) $\mu = \int_0^1 x\left(e - e^x\right)dx = \int_0^1 \left(ex - xe^x\right)dx$ *(M1)*

$= \left[\frac{ex^2}{2}\right]_0^1 - \int_0^1 xe^x\,dx = \frac{e}{2} - \left[xe^x - e^x\right]_0^1$ *(M1)*

$= \frac{e}{2} - 1$ *(A1)*

Variance $= \int_0^1 x^2\left(e - e^x\right)dx - \left(\frac{e}{2} - 1\right)^2$ *(M1)*

$= \left[\frac{ex^3}{3} - e^x\left(x^2 - 2x + 2\right)\right]_0^1 - \left(\frac{e}{2} - 1\right)^2$ *(M1)*

$= 2 - \frac{2}{3}e - \frac{e^2}{4} + e - 1$

$= 1 + \frac{e}{3} - \frac{e^2}{4}$ *(A1)*

[6 marks]

(d) p(battery lasts more than 6 months) $= p\left(x > \frac{1}{2}\right)$

$= \int_{1/2}^1 \left(e - e^x\right)dx$ *(M1)*

$= \left[ex - e^x\right]_{1/2}^1$

$= \sqrt{e} - \frac{e}{2}$ or 0.290 (3 s.f.) *(A1)*

[2 marks]

(e) p(no battery failed) $= p$(all lasted more than 6 months) *(M1)*

$= \left(\sqrt{e} - \frac{e}{2}\right)^3$ or 0.0243 (3 s.f.) *(A1)*

[2 marks]

(f) p(exactly one battery failed) $= \binom{3}{2}\left(1 - \sqrt{e} + \frac{e}{2}\right)\left(\sqrt{e} - \frac{e}{2}\right)^2$ *(M1)*

≈ 0.179 (3 s.f.) *(A1)*

[2 marks]

Total [17 marks]

5. (i) (a) $(3*9)*13 \quad = 13*13 \quad = 1$ *(M1)*

and $3*(9*13) \quad = 3*5 \quad = 1$ *(M1)*

hence $(3*9)*13 \quad = 3*(9*13)$ *(AG)*

[2 marks]

(b) To show that $(U, *)$ is a group we need to show that:

(1) U is closed under $*$.

A table is an easy way of showing closure for this finite set.

*	1	3	5	9	11	13
1	1	3	5	9	11	13
3	3	9	1	13	5	11
5	5	1	11	3	13	9
9	9	13	3	11	1	5
11	11	5	13	1	9	3
13	13	11	9	5	3	1

(C4)

Note: Award *(C4)* for a completely accurate table, *(C3)* for 1 or 2 errors, *(C2)* for 3 or 4 errors, *(C1)* for 5 or 6 errors, *(C0)* for 7 or more errors.

since for each $a, b \in U$, $a * b \in U$, **closure** is shown. *(C1)*

(2) Since multiplication is **associative**, it is true in this case too. *(C2)*

(3) Since $1*a = a*1 = a$ for all $a \in U$, 1 is the **identity**. *(C2)*

(4) 1 appears in each row of the table once, so every element has a unique **inverse**.
$(1^{-1} = 1, 3^{-1} = 5, 5^{-1} = 3, 9^{-1} = 11, 11^{-1} = 9, 13^{-1} = 13)$ *(C2)*

[11 marks]

(c) (i) If G is a group and if there exists $a \in G$, such that

$$G = \{a^n : n \in \mathbb{Z}\}$$

Then G is a cyclic group and a is called a generator. *(C2)*

[2 marks]

(ii) By inspection:

3 is a generator since:

$3^2 = 9, 3^3 = 13, 3^4 = 11$ *(M1)*

$3^5 = 5, 3^6 = 1$ *(A1)*

Also, 5 is a generator:

$5^2 = 11, 5^3 = 13, 5^4 = 9$ *(M1)*

$5^5 = 3, 5^6 = 1$ *(A1)*

9 cannot be a generator since

$9^3 = 1$ *(C1)*

similarly $11^3 = 1$ *(C1)*

and $13^2 = 1$. *(C1)*

[7 marks]

continued...

Question 5 (i) continued

(d) Since the order of this group is 6, by Lagrange's Theorem, the proper subgroups can only have orders 2 or 3. ***(R1)***

Since 13 is the only self inverse $13^2 = 1$, ***(R1)***
the only subgroup of order 2 is $\{1, 13\}$ ***(A1)***
No sub-group may include 3 or 5 since these are the generators of the group.
The only elements left are 9 and 11. ***(R1)***
Now, $9 * 11 = 1, 9^2 = 11$, and $11^2 = 9$. ***(M2)***
Therefore, $\{1, 9, 11\}$ is the other sub-group. ***(A1)***

[7 marks]

(ii) (a) Since $\forall a \in G, e \circ a = a \circ e$ because e is the identity element of the group. ***(R2)***
Then $e \in H$. ***(AG)***

[2 marks]

(b) Let $x, y \in H$, then

$(x \circ y) \circ a = x \circ (y \circ a)$	(by associativity)	***(R1)***
$= x \circ (a \circ y)$	(since $y \in H$)	***(R1)***
$= (x \circ a) \circ y$	(associativity)	***(R1)***
$= (a \circ x) \circ y$	($x \in H$)	***(R1)***
$= a \circ (x \circ y)$	(associativity)	***(R1)***

Therefore, $(x \circ y) \circ a = a \circ (x \circ y)$
$\Rightarrow (x \circ y) \in H$. ***(AG)***

[5 marks]

(c)

	$e \circ a = a \circ e$	identity	***(R1)***
$\Rightarrow$	$(x^{-1} \circ x) \circ a = a \circ (x^{-1} \circ x)$		***(R1)***
$\Rightarrow$	$x^{-1} \circ (x \circ a) = (a \circ x^{-1}) \circ x$	associativity	
$\Rightarrow$	$x^{-1} \circ (a \circ x) = (a \circ x^{-1}) \circ x$	$x \in H$	***(R1)***
$\Rightarrow$	$(x^{-1} \circ a) \circ x = (a \circ x^{-1}) \circ x$	associativity	***(R1)***
Therefore,	$x^{-1} \circ a = a \circ x^{-1}$	cancellation law	

and $x^{-1} \in H$ ***(AG)***

[4 marks]

Total [40 marks]

6. (i) (a)

	A	B	C	D	E	F
A	0	1	2	2	2	1
B	1	0	1	2	3	2
C	2	1	0	1	2	1
D	2	2	1	0	2	1
E	2	3	2	2	0	1
F	1	2	1	1	1	0

(A4)

Note: Award *(A4)* for a completely correct table, *(A3)* for 1 error, *(A2)* for 2 errors, *(A1)* for 3 errors, *(A0)* for 4 or more errors.

[4 marks]

(b) (i) The index for each city is given below:

A: $\frac{8}{2} = 4$; B: $\frac{9}{2} = 4.5$

C: $\frac{7}{3} = 2.\overline{3}$; D: $\frac{8}{2} = 4$

E: $\frac{10}{1} = 10$; F: $\frac{6}{4} = 1.5$ *(C2)*

Note: Award *(C2)* for all correct, *(C1)* for 1 error, *(C0)* for 2 or more errors.

City F is the most accessible since its index is 1.5. *(C1)*

City E is the least accessible since its index is 10. *(C1)*

(ii) The indices are now:

A: $2.\overline{3}$ B: 4.5 C: 1.5

D: 4 E: 10 F: 1.5 *(M1)*

C and F are the most accessible cities. *(C1)*
E is still the least accessible. *(C1)*

[7 marks]

continued...

Question 6 continued

(ii) (a)

$$\begin{array}{c|c} \text{U} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(A3)
[3 marks]

Note:	Award *(A3)* for a completely correct table, *(A2)* for 1 or 2 errors, *(A1)* for 3 or 4 errors, *(A0)* for 5 or more errors.

(b) Adjacency matrix for V.

$$\begin{array}{c|c} & \begin{matrix} A & E & B & F & C & G & D & H \end{matrix} \\ \begin{matrix} A \\ E \\ B \\ F \\ C \\ G \\ D \\ H \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(M2)(A2)

Note:	Award *(M2)* for correct correspondence, *(M1)* for 1 or 2 errors, *(M0)* for 3 or more errors. Award *(A2)* for correct matrix, *(A1)* for 1 or 2 errors, *(A0)* for 3 or more errors.

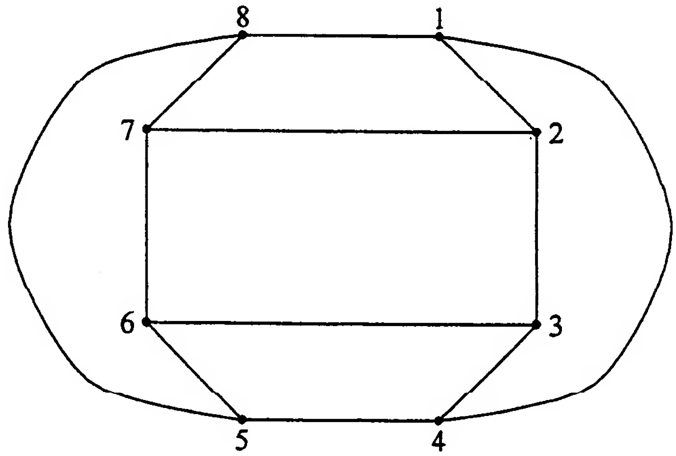
Since there is a one-to-one correspondence in which 2 vertices are adjacent if and only if their images are adjacent, and since the matrix structures are identical, then the two graphs are isomorphic. *(R1)* *(R1)*

[6 marks]

continued...

Question 6 (ii) continued

(c) By redrawing U as shown below we can see that it is planar and since U ≅ V, V is also planar as isomorphism preserves planarity. *(C1)(R1)* *(A1)(C1)*



U

[4 marks]

(iii) (a)

Vertices added to the Tree	Edge added	Weight
3	∅	0
5	3,5	10
6	3,6	20
7	5,7	30
10	6,10	30
1	3,1	40
2	1,2	30
11	2,11	30
9	1,9	40
4	6,4	40
8	7,8	40
		310

(R2)(A4)(M1)

(A1)

Note:	Award *(R2)* for correct algorithms, *(R1)* for 1 error, *(R0)* for 2 or more errors. Award *(A4)* for correct calculations, *(A3)* for 1 error, *(A2)* for 2 errors, *(A1)* for 3 errors, *(A0)* for 4 or more errors. Award *(M1)* for tree/table/method. Award *(A1)* for minimum weight.

[8 marks]

continued…

Question 6 (iii) continued

(b)

Vertices and potentials

Step	Starting Vertex	1	2	3	4	5	6	7	8	9	10	11
1	1		$\boxed{6}$	$\boxed{3}$	8							
2	2				13	20	22					
3	3				$\boxed{5}$	15	22					
4	4					$\boxed{11}$		$\boxed{29}$				
5	5						$\boxed{18}$	31	34	67		
6	6							47	$\boxed{33}$	$\boxed{48}$		
7	7									70	71	
8	8									83	$\boxed{58}$	
9	9										88	$\boxed{80}$
10	10											$\boxed{80}$

(R2)(A4)

Minimum = 80 ***(A1)***

Any of two paths: 1 - 3 - 4 - 5 - 6 - 8 - 10 - 11 or 1 - 3 - 4 - 5 - 6 - 9 - 11 ***(A1)***

Note: Award ***(R2)*** for **correct algorithms**, ***(R1)*** for 1 error, such as an unclear statement or application, ***(R0)*** for 2 or more errors.
Award ***(A4)*** for correct calculations, ***(A3)*** for 1 error, ***(A2)*** for 2 errors, ***(A1)*** for 3 errors, ***(A0)*** for 4 or more errors.

[8 marks]

Total [40 marks]

7. (i) (a) $p(X > 327)$

$= p\left(Z > \frac{327-330}{6}\right)$ *(M1)*

$= p\left(Z > -\frac{1}{2}\right) = 0.6915$

$= 0.692$ (3 s.f.) *(A1)*

[2 marks]

(b) Let x_0 be the required amount of coffee.

$p(X > x_0) = 0.95$ *(M1)*

$\Rightarrow \frac{x_0 - 330}{6} = -1.645$ *(M1)*

$\Rightarrow x_0 = 330 - 1.645 \times 6 = 320.13$

$= 320$ (3 s.f.) *(A1)*

[3 marks]

(c) $p(\overline{X} \geq 332) = p\left(Z \geq \frac{332-330}{\frac{6}{\sqrt{60}}}\right)$ *(M1)*

$= p(Z \geq 2.58)$ *(M1)*

$= 1 - 0.9951$ or $1 - 0.99509$ (from GDC)

$= 0.0049$ or 0.00491 (3 s.f.) *(A1)*

[3 marks]

(d) Since a sample with this mean has a chance of 0.5 % of being produced by this machine, we may agree with the vendor that the true mean of the volume of coffee seems to be higher than 330 ml. *(R1)* *(R1)* *(R1)*

[3 marks]

(e) A 95 % confidence interval is:

$332.78 \pm 1.96 \times \frac{6}{\sqrt{25}}$ *(M2)*

$= (330.43, 335.13)$ *(A2)*

[4 marks]

(f) Since $330 \notin (330.43, 335.13)$ we can conclude that there is enough evidence to support the hypothesis. *(M1)* *(R1)*

[2 marks]

(g) Among the 25 cups sampled, there are 18 that have more than 330 ml. *(M1)*

Let p be the proportion of cups that have more coffee than the vector desires.

An estimate for p is $\frac{18}{25} = 0.72$. *(A1)*

A 95 % confidence interval for p is

$0.72 \pm 1.96\sqrt{\frac{0.72 \times 0.28}{25}}$ *(M1)*

$= (0.544, 0.896)$ *(A1)*

[4 marks]

continued...

Question 7 continued

(ii) (a) To calculate expected frequencies, we multiply 4000 by the probability of each cell:

$$p(80.5 \le X \le 90.5) = p\left(\frac{80.5-100}{10} \le Z \le \frac{90.5-100}{10}\right) \qquad \textbf{\textit{(M1)}}$$
$$= p(-1.95 \le Z \le -0.95)$$
$$= 0.1711 - 0.0256$$
$$= 0.1455$$

Therefore, the expected frequency $= 4000 \times 0.1455$ ***(M1)***

≈ 582 ***(A1)***

Similarly:

$$p(90.5 \le X \le 100.5) = 0.5199 - 0.1711$$
$$= 0.3488$$

Frequency $= 4000 \times 0.3488$

≈ 1396 ***(A1)***

And

$$p(100.5 \le X \le 110.5) = 0.8531 - 0.5199$$
$$= 0.3332$$

Frequency $= 4000 \times 0.3332$

≈ 1333 ***(A1)***

[5 marks]

(b) To test the goodness of fit of the normal distribution, we use the χ^2 distribution. Since the last cell has an expected frequency less than 5, it is combined with the cell preceding it. There are therefore $7 - 1 = 6$ degrees of freedom. ***(C1)***

$$\chi^2 = \frac{(20-6)^2}{6} + \frac{(90-96)^2}{96} + \frac{(575-582)^2}{582} + \frac{(1282-1396)^2}{1396}$$

$$+ \frac{(1450-1333)^2}{1333} + \frac{(499-507)^2}{507} + \frac{(84-80)^2}{80} \qquad \textbf{\textit{(M1)}}$$

$= 53.03$ ***(A1)***

H_0: Distribution is Normal with $\mu = 100$ and $\sigma = 10$.
H_1: Distribution is not Normal with $\mu = 100$ and $\sigma = 10$. ***(M1)***

$\chi^2_{(0.95,6)} = 14.07$

Since $\chi^2 = 53.0 > \chi^2_{\text{critical}} = 14.07$, we reject H_0. ***(A1)***

Therefore, we have enough evidence to suggest that the normal distribution with mean 100 and standard deviation 10 does not fit the data well. ***(R1)***

[6 marks]

Note:	If a candidate has not combined the last 2 cells, award ***(C0)(M1)(A0)(M1)(A1)(R1)*** (or as appropriate).

continued...

Question 7 continued

(iii) This is the case for a contingency table. The observed frequencies are given. The expected frequencies can be calculated by multiplying the column and row totals for each cell and dividing the product by the total number of observations,

e.g. First cell: $\dfrac{40\times 80}{100}=32$. (Others are calculated in a similar manner.) *(M1)*

Expected frequencies	No flu immunisation injections	Flu immunisation injections
Cold	32	8
No cold	48	12

(A1)

Since there is only $(2-1)(2-1)=1$ degree of freedom we use Yates' continuity correction factor: *(R1)*

$$\chi^2=\frac{(|35-32|-0.5)^2}{32}+\frac{(|5-8|-0.5)^2}{8}+\frac{(|45-48|-0.5)^2}{48}+\frac{(|15-12|-0.5)^2}{12}$$ *(M1)*

$=1.628$ *(A1)*

H_0: Flu immunisation injections and suffering from colds are independent.
H_1: There is evidence of dependence between flu immunisation injections and colds. *(M1)*

$\chi^2_{(0.95,1)} = 3.84.$

Since $\chi^2_{\text{test}}=1.628<\chi^2_{\text{critical}}=3.84$. *(A1)*

We do not have enough evidence to reject H_0. Hence we do not have enough evidence to support the claim that flu injections help reduce the number of people suffering from colds. *(R1)*

[8 marks]

Total [40 marks]

8. (i) (a) $f(1) = -3$ *(A1)*

$f(2) = 2$ *(A1)*

[2 marks]

(b) $f'(x) = 6x^2 - 30x + 36$ *(C1)*

[1 mark]

(c) Since $f(1) = -3$, and $f(2) = 2$, by the intermediate value theorem, $f(x)$ has at least one zero between 1 and 2. *(M1)* *(R1)*

Since $f'(x) > 0$ in the interval $1 < x < 2$, $f(x)$ can intersect x-axis only once. *(R1)*

[3 marks]

(d) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ *(M1)*

$x_1 = 1.5 - \dfrac{f(1.5)}{f'(1.5)} = 1.2778$ *(M1)*

$x_2 = 1.3204$ *(A1)*

$x_3 = 1.3223$

$x_4 = 1.3223 \Rightarrow$

$x \approx 1.322$ (3 d.p.) *(A1)*

[4 marks]

(e) $2x^3 = 15x^2 - 36x + 26$

$x = \sqrt[3]{\dfrac{1}{2}\left(15x^2 - 36x + 26\right)}$ *(M1)*

$x_{n+1} = \sqrt[3]{\dfrac{1}{2}\left(15x_n^2 - 36x_n + 26\right)}$ *(C1)*

$x_1 = \sqrt[3]{2.875} = 1.4219$ *(A1)*

$x_2 = 1.3697$

$x_3 = 1.34$ (3 s.f.) *(A1)*

[4 marks]

(ii) (a) $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^2\, 2^{n+2}}{3^{n+1}} \times \dfrac{3^n}{n^2\, 2^{n+1}}$ *(M1)*

$= \lim\limits_{n\to\infty} \dfrac{2(n+1)^2}{3n^2}$ *(M1)*

$= \dfrac{2}{3} < 1$ *(A1)*

This series converges by the ratio test since $\lim \dfrac{a_{n+1}}{a_n} < 1$. *(R1)*

[4 marks]

continued...

Question 8 (ii) continued

(b) Since $f(x)=\dfrac{x^{k-1}}{x^k+k}$ is decreasing and continuous for $x>2,$ the integral test may be applied. ***(R1)***

$$\int_1^{\infty}\frac{x^{k-1}}{x^k+k}\mathrm{d}x=\lim_{a\to\infty}\left[\frac{1}{k}\ln\left(x^k+k\right)\right]_1^a \quad \textit{(M2)}$$

$$=\lim_{a\to\infty}\left[\frac{1}{k}\ln\left(a^k+k\right)-\frac{1}{k}\ln(k+1)\right] \quad \textit{(A1)}$$

$$=\infty$$

Thus the integral diverges and consequently the series diverges too. ***(R1)***

[5 marks]

(iii) (a) $f'(x)=\dfrac{-x}{\sqrt{2-x^2}}$ ***(A1)***

$$f''(x)=\frac{-\sqrt{2-x^2}+x\left(\frac{1}{2}\right)(-2x)\left(2-x^2\right)^{-1/2}}{2-x^2} \quad \textit{(M1)}$$

$$=\frac{-2+x^2-x^2}{\left(2-x^2\right)\sqrt{2-x^2}} \quad \textit{(A1)}$$

$$=\frac{2}{\left(x^2-2\right)\sqrt{2-x^2}} \quad \textit{(AG)}$$

[3 marks]

(b) $f(x)\approx f(0)+f'(0)x+f''(0)\dfrac{x^2}{2!}+\ldots+\dfrac{f^{(n)}(0)}{n!}x^n+\ldots$ ***(M1)***

$$f(0)=\sqrt{2}$$
$$f'(0)=0$$
$$f''(0)=-\frac{\sqrt{2}}{2}$$
$$f'''(0)=0$$
$$f^4(0)=-\frac{3\sqrt{2}}{4} \quad \textit{(A2)}$$

$$\Rightarrow\sqrt{2-x^2}\approx\sqrt{2}-\left(\frac{\sqrt{2}}{2}\right)\frac{x^2}{2}-\frac{3\sqrt{2}}{4}\left(\frac{x^4}{24}\right) \quad \textit{(M1)}$$

$$=\sqrt{2}-\frac{\sqrt{2}}{4}x^2-\frac{\sqrt{2}}{32}x^4=\sqrt{2}\left(1-\frac{x^2}{4}-\frac{1}{32}x^4\right)$$

$$\Rightarrow k=-\frac{1}{32} \quad \textit{(A1)}$$

[5 marks]

continued...

Question 8 (iii) continued

(c) $\int_0^1 x^2\sqrt{2-x^2}\,dx \approx \int_0^1 \sqrt{2}\left(x^2 - \frac{x^4}{4} - \frac{x^6}{32}\right)dx$ *(M1)*

$= \sqrt{2}\left[\frac{x^3}{3} - \frac{x^5}{20} - \frac{x^7}{224}\right]_0^1$ *(M2)*

$= \sqrt{2}\left(\frac{1}{3} - \frac{1}{20} - \frac{1}{224}\right)$ *(A1)*

$= 0.39438$

(AG)
[4 marks]

(d) $x = \sqrt{2}\sin\theta \Rightarrow \sin\theta = \frac{x}{\sqrt{2}}$

$\Rightarrow x = 0, \theta = 0$ or $x = 1, \theta = \frac{\pi}{4}$

$dx = \sqrt{2}\cos\theta\, d\theta$
$\sqrt{2-x^2} = \sqrt{2}\cos\theta$ *(M1)*

Therefore, $\int_0^1 x^2\sqrt{2-x^2}\,dx = \int_0^{\pi/4} 4\sin^2\theta\cos^2\theta\, d\theta$ *(M1)*

$= \int_0^{\pi/4} \sin^2 2\theta\, d\theta$

$= \frac{1}{2}\int_0^{\pi/4}(1-\cos 4\theta)\,d\theta$ *(M1)*

$= \frac{1}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/4}$

$= \frac{\pi}{8}$ *(A1)*

Therefore, $\frac{\pi}{8} \approx 0.39438 \Rightarrow \pi \approx 3.1550$ *(A1)*

[5 marks]

Total [40 marks]